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Conditional symmetry and new classical solutions of the Yang–Mills equations

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Abstract. We suggest an effective method for reducing the Yang–Mills equations to systems of ordinary differential equations. With the use of this method we construct extensive families of new exact solutions of the Yang–Mills equations. Analysis of the solutions thus obtained shows that they correspond to the conditional (non-classical) symmetry of the equations under study.

1. Introduction

A majority of papers devoted to the construction of the explicit form of the exact solutions of $SU(2)$ Yang–Mills equations (YMEs)

$$\partial_\nu \partial^\nu \boldsymbol{A}_\mu - \partial^\mu \partial_\nu \boldsymbol{A}_\nu + e\Big((\partial_\nu \boldsymbol{A}_\nu) \times \boldsymbol{A}_\mu - 2(\partial_\nu \boldsymbol{A}_\mu) \times \boldsymbol{A}_\nu + (\partial^\mu \boldsymbol{A}_\nu) \times \boldsymbol{A}^\nu \Big)$$
$$+ e^2 \boldsymbol{A}_\nu \times (\boldsymbol{A}^\nu \times \boldsymbol{A}_\mu) = \boldsymbol{0} \qquad (1)$$

are based on the ansätze for the Yang–Mills field $\boldsymbol{A}_\mu(x)$ suggested by Wu and Yang, Rosen, 't Hooft, Corrigan and Fairlie, Wilczek, Witten (see [1] and references therein). There were further developments for the self-dual YMEs (which form the first-order system of nonlinear partial differential equations such that the system (1) is its differential consequence). Let us mention here the Atiyah–Hitchin–Drinfeld–Manin method for obtaining instanton solutions [2] and its generalization due to Nahm. However, the solution set of the self-dual YMEs is only a subset of the solutions of the YMEs (1) and the problem of constructing new non-self-dual solutions of the system (1) is, in fact, completely open (see, also [1]). As the development of new approaches to the construction of exact solutions of YMEs is a very interesting mathematical problem, it may also be of importance for physics. The reason is that all famous mathematical models of elementary particles such as solitons, instantons, merons are quite simply particular solutions of some nonlinear partial differential equations.

A natural approach to the construction of particular solutions of YMEs (1) is to utilize their symmetry properties in the same way as used in [9, 10, 16] (see also [15], where the reduction of the Euclidean self-dual YMEs is considered). The apparatus of the theory of

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Lie transformation groups makes it possible to reduce the system of partial differential equations (PDEs) (1) to systems of nonlinear ordinary differential equations (ODEs) by using special ansätze (invariant solutions) [10, 18, 20]. If one succeeds in constructing general or particular solutions of the said ODEs (which is an extremely difficult problem), then on substituting the results in the corresponding ansätze one gets exact solutions of the initial system of PDEs (1).

Another possibility for constructing exact solutions of YMEs is to use their conditional (non-Lie) symmetry (for more details about the conditional symmetry of the equations of mathematical physics, see [6, 8] and also [10, 12]) which has much in common with the 'non-classical symmetry' of PDEs of Bluman and Cole [3] (see also [17, 19]) and the 'direct method of reduction of PDEs' of Clarkson and Kruskal [4]. But the prospects for a systematic and exhaustive study of the conditional symmetry of a system of twelve second-order nonlinear PDEs (1) seem to be rather remote. It should be said that so far there is no complete description of the conditional symmetry of the nonlinear wave equation even in the case of one space variable.

A principal idea of the method of ansätze, as well as of the direct method of reduction of PDEs, is a special choice of the class of functions to which the a possible solution should belong. Within the framework of the above methods, a solution of the system (1) is sought in the form

$$A_\mu = H_\mu\Big(x, B_\nu(\omega(x))\Big) \qquad \mu = \overline{0,3}$$

where H_μ are smooth functions chosen in such a way that substitution of the above expressions in the Yang–Mills equations results in a system of ODEs for 'new' unknown vector functions B_ν of one variable ω. However, the problem of reduction of YMEs posed in this way seemed to be hopeless. Really, if we restrict ourselves to the case of a linear dependence of the above ansatz on B_ν

$$A_\mu(x) = R_{\mu\nu}(x)B^\nu(\omega) \tag{2}$$

where $B_\nu(\omega)$ are new unknown vector functions, $\omega = \omega(x)$ is a new independent variable, then a requirement for reduction of (1) to a system of ODEs by virtue of (2) gives rise to a system of nonlinear PDEs for 17 unknown functions $R_{\mu\nu}$, ω. What is more, the system obtained is in no way simpler than the initial Yang–Mills equations (1). It means that some additional information about the structure of the matrix function $R_{\mu\nu}$ should be input into the ansatz (2). This can be done in various ways. But the most natural one is to use the information about the structure of solutions provided by the Lie symmetry of the equation under study.

In [11] we suggest an effective approach to the study of the conditional symmetry of the nonlinear Dirac equation based on its Lie symmetry. We have observed that all Poincaré-invariant ansätze for the Dirac field $\psi(x)$ can be represented in a unified form by introducing several arbitrary elements (functions) $u_1(x)$, $u_2(x), \ldots, u_N(x)$. As a result, we get an ansatz for the field $\psi(x)$ which reduces the nonlinear Dirac equation to a system of ODEs provided the functions $u_i(x)$ satisfy some compatible over-determined system of nonlinear PDEs. After integrating it, we obtain a number of new ansätze that cannot in principle be obtained within the framework of the classical Lie approach.

In the present paper we will demonstrate that the same idea proves to be fruitful for obtaining new (non-Lie) reductions of YMEs and for constructing new exact solutions of the system (1).

2. Reduction of YMEs

In [16] we obtained a complete list of $P(1,3)$-inequivalent ansätze for the Yang–Mills field which are invariant under the three-parameter subgroups of the Poincaré group $P(1,3)$. Analysing these ansätze we come to conclusion that they can be represented in the unified form (2), where $\boldsymbol{B}_\nu(\omega)$ are new unknown vector functions, $\omega = \omega(x)$ is a new independent variable and functions $R_{\mu\nu}(x)$ are given by the expressions

$$\begin{aligned} R_{\mu\nu}(x) = {} & (a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (a_\mu d_\nu - d_\mu a_\nu)\sinh\theta_0 + 2(a_\mu + d_\mu) \\ & \times[(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu \\ & + (\theta_1^2 + \theta_2^2)e^{-\theta_0}(a_\nu + d_\nu)] - (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3 \\ & -(c_\mu b_\nu - b_\mu c_\nu)\sin\theta_3 - 2e^{-\theta_0}(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu). \end{aligned} \tag{3}$$

In (3) $\theta_\mu(x)$ are some smooth functions and what is more $\theta_a = \theta_a(\xi,\ b_\mu x^\mu,\ c_\mu x^\mu)$, $a = 1, 2$; $\xi = \frac{1}{2}k_\mu x^\mu = \frac{1}{2}(a_\mu x^\mu + d_\mu x^\mu)$; a_μ, b_μ, c_μ, d_μ are arbitrary constants satisfying the following relations:

$$a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1$$

$$a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0.$$

Hereafter, summation over the repeated indices from 0 to 3 is understood. Raising and lowering of the indices is performed with the help of the tensor $g_{\mu\nu} = \mathrm{diag}(1,\ -1,\ -1,\ -1)$, e.g. $R^\alpha_\mu = g_{\alpha\beta}R_{\beta\mu}$.

A choice of the functions $\omega(x)$, $\theta_\mu(x)$ is determined by the requirement that substitution of the ansatz (2) in the YMEs yields a system of ODEs for the vector function $\boldsymbol{B}_\mu(\omega)$.

By a direct check one can convince oneself that the following assertion holds true.

Lemma. The ansatz (2), (3) reduces the YMEs (1) to the system of ODEs iff the functions $\omega(x)$, $\theta_\mu(x)$ satisfy the following system of PDEs:

$$\omega_{x_\mu}\omega_{x^\mu} = F_1(\omega) \tag{4a}$$

$$\Box\omega = F_2(\omega) \tag{4b}$$

$$R_{\alpha\mu}\omega_{x_\alpha} = G_\mu(\omega) \tag{4c}$$

$$R_{\alpha\mu x_\alpha} = H_\mu(\omega) \tag{4d}$$

$$R^\alpha_\mu R_{\alpha\nu x_\beta}\omega_{x^\beta} = Q_{\mu\nu}(\omega) \tag{4e}$$

$$R^\alpha_\mu \Box R_{\alpha\nu} = S_{\mu\nu}(\omega) \tag{4f}$$

$$R^\alpha_\mu R_{\alpha\nu x_\beta}R_{\beta\gamma} + R^\alpha_\nu R_{\alpha\gamma x_\beta}R_{\beta\mu} + R^\alpha_\gamma R_{\alpha\mu x_\beta}R_{\beta\nu} = T_{\mu\nu\gamma}(\omega) \tag{4g}$$

where F_1, F_2, $G_\mu, \ldots, T_{\mu\nu\gamma}$ are some smooth functions, $\mu, \nu, \gamma = \overline{0,3}$. And what is more, a reduced equation has the form

$$\begin{aligned} & k_{\mu\gamma}\ddot{\boldsymbol{B}}^\gamma + l_{\mu\gamma}\dot{\boldsymbol{B}}^\gamma + m_{\mu\gamma}\boldsymbol{B}^\gamma + eq_{\mu\nu\gamma}\dot{\boldsymbol{B}}^\nu \times \boldsymbol{B}^\gamma + eh_{\mu\nu\gamma}\boldsymbol{B}^\nu \times \boldsymbol{B}^\gamma \\ & \quad + e^2\boldsymbol{B}_\gamma \times (\boldsymbol{B}^\gamma \times \boldsymbol{B}_\mu) = \mathbf{0} \end{aligned} \tag{5}$$

where

$$
\begin{aligned}
k_{\mu\gamma} &= g_{\mu\gamma}F_1 - G_\mu G_\gamma \\
l_{\mu\gamma} &= g_{\mu\gamma}F_2 + 2Q_{\mu\gamma} - G_\mu H_\gamma - G_\mu \dot{G}_\gamma \\
m_{\mu\gamma} &= S_{\mu\gamma} - G_\mu \dot{H}_\gamma \\
q_{\mu\nu\gamma} &= g_{\mu\gamma}G_\nu + g_{\nu\gamma}G_\mu - 2g_{\mu\nu}G_\gamma \\
h_{\mu\nu\gamma} &= \tfrac{1}{2}(g_{\mu\gamma}H_\nu - g_{\mu\nu}H_\gamma) - T_{\mu\nu\gamma}.
\end{aligned}
\tag{6}
$$

Thus, to describe all ansätze of the form (2), (3) reducing the YMEs to a system of ODEs, one has to construct the general solution of the over-determined system of PDEs (3), (4). Let us emphasize that the system (3), (4) is compatible, since the ansätze for the Yang–Mills field $A_\mu(x)$ invariant under the three-parameter subgroups of the Poincaré group satisfy equations (3), (4) with some specific choice of the functions $F_1,\ F_2, \ldots, T_{\mu\nu\gamma}$ [16].

Integration of the system of nonlinear PDEs (3), (4) demands a huge number of computations. That is why we present here only the principal idea of our approach to solving the system (3), (4). When integrating it we essentially use the fact that the general solution of the system of equations (4*a*), (4*b*) is known [13]. With $\omega(x)$ already known we proceed to integration of the linear PDEs (4*c*), (4*d*). Next, we substitute the results obtained in the remaining equations (4) and get the final form of the functions $\omega(x)$, $\theta_\mu(x)$.

Before presenting the results of integration of the system of PDEs (3), (4) we make a remark. As the direct check shows, the structure of the ansatz (2), (3) is not altered by the change of variables

$$
\begin{aligned}
&\omega \to \omega' = T(\omega) \qquad \theta_0 \to \theta_0' = \theta_0 + T_0(\omega) \\
&\theta_1 \to \theta_1' = \theta_1 + e^{\theta_0}\Big(T_1(\omega)\cos\theta_3 + T_2(\omega)\sin\theta_3\Big) \\
&\theta_2 \to \theta_2' = \theta_2 + e^{\theta_0}\Big(T_2(\omega)\cos\theta_3 - T_1(\omega)\sin\theta_3\Big) \\
&\theta_3 \to \theta_3' = \theta_3 + T_3(\omega)
\end{aligned}
\tag{7}
$$

where $T(\omega)$, $T_\mu(\omega)$ are arbitrary smooth functions. That is why solutions of the system (3), (4) connected by the relations (7) are considered as equivalent.

Integrating the system of PDEs within the above equivalence relations, we obtain the set of ansätze containing the ones equivalent to the Poincaré-invariant ansätze. We list below the corresponding expressions for the functions θ_μ, ω:

$$\theta_\mu = 0 \qquad \omega = d\cdot x \tag{8a}$$

$$\theta_\mu = 0 \qquad \omega = a\cdot x \tag{8b}$$

$$\theta_\mu = 0 \qquad \omega = k\cdot x \tag{8c}$$

$$\theta_0 = -\ln|k\cdot x| \qquad \theta_1 = \theta_2 = 0 \qquad \theta_3 = \alpha\ln|k\cdot x| \qquad \omega = (a\cdot x)^2 - (d\cdot x)^2 \tag{8d}$$

$$\theta_0 = -\ln|k\cdot x| \qquad \theta_1 = \theta_2 = \theta_3 = 0 \qquad \omega = c\cdot x \tag{8e}$$

$$\theta_0 = -b\cdot x \qquad \theta_1 = \theta_2 = \theta_3 = 0 \qquad \omega = c\cdot x \tag{8f}$$

$$\theta_0 = -b\cdot x \qquad \theta_1 = \theta_2 = \theta_3 = 0 \qquad \omega = b\cdot x - \ln|k\cdot x| \tag{8g}$$

$$
\begin{aligned}
&\theta_0 = \alpha\arctan(b\cdot x/c\cdot x) \qquad \theta_1 = \theta_2 = 0 \\
&\theta_3 = -\arctan(b\cdot x/c\cdot x) \qquad \omega = (b\cdot x)^2 + (c\cdot x)^2
\end{aligned}
\tag{8h}
$$

$$\theta_0=\theta_1=\theta_2=0 \qquad \theta_3=-a\cdot x \qquad \omega=d\cdot x \tag{8i}$$

$$\theta_0=\theta_1=\theta_2=0 \qquad \theta_3=d\cdot x \qquad \omega=a\cdot x \tag{8j}$$

$$\theta_0=\theta_1=\theta_2=0 \qquad \theta_3=-\tfrac{1}{2}k\cdot x \qquad \omega=a\cdot x-d\cdot x \tag{8k}$$

$$\theta_0=0 \qquad \theta_1=\tfrac{1}{2}(b\cdot x-\alpha c\cdot x)(k\cdot x)^{-1} \qquad \theta_2=\theta_3=0 \qquad \omega=k\cdot x \tag{8l}$$

$$\theta_0=\theta_2=\theta_3=0 \qquad \theta_1=\tfrac{1}{2}c\cdot x \qquad \omega=k\cdot x \tag{8m}$$

$$\theta_0=\theta_2=\theta_3=0 \qquad \theta_1=-\tfrac{1}{4}k\cdot x \qquad \omega=4b\cdot x+(k\cdot x)^2 \tag{8n}$$

$$\theta_0=\theta_2=\theta_3=0 \qquad \theta_1=-\tfrac{1}{4}k\cdot x \qquad \omega=4(\alpha b\cdot x-c\cdot x)+\alpha(k\cdot x)^2 \tag{8o}$$

$$\begin{aligned}&\theta_0=-\ln|k\cdot x| \qquad \theta_1=\theta_2=0\\&\theta_3=-\arctan(b\cdot x/c\cdot x) \qquad \omega=(b\cdot x)^2+(c\cdot x)^2\end{aligned} \tag{8p}$$

$$\begin{aligned}&\theta_0=\theta_3=0 \qquad \theta_1=\tfrac{1}{2}\Big(c\cdot x+(\alpha+k\cdot x)b\cdot x\Big)\Big(1+k\cdot x(\alpha+k\cdot x)\Big)^{-1}\\&\theta_2=-\tfrac{1}{2}(b\cdot x-c\cdot xk\cdot x)\Big(1+k\cdot x(\alpha+k\cdot x)\Big)^{-1} \qquad \omega=k\cdot x\end{aligned} \tag{8q}$$

$$\begin{aligned}&\theta_0=-\ln|k\cdot x| \qquad \theta_1=\tfrac{1}{2}b\cdot x(k\cdot x)^{-1}\\&\theta_2=\theta_3=0 \qquad \omega=(a\cdot x)^2-(b\cdot x)^2-(d\cdot x)^2\end{aligned} \tag{8r}$$

$$\theta_0=-\ln|k\cdot x| \qquad \theta_1=\tfrac{1}{2}b\cdot x(k\cdot x)^{-1} \qquad \theta_2=\theta_3=0 \qquad \omega=c\cdot x \tag{8s}$$

$$\begin{aligned}&\theta_0=-\ln|k\cdot x| \qquad \theta_1=\tfrac{1}{2}b\cdot x(k\cdot x)^{-1}\\&\theta_2=\theta_3=0 \qquad \omega=\ln|k\cdot x|-c\cdot x\end{aligned} \tag{8t}$$

$$\begin{aligned}&\theta_0=-\ln|k\cdot x| \qquad \theta_1=\tfrac{1}{2}(b\cdot x-\ln|k\cdot x|)(k\cdot x)^{-1}\\&\theta_2=\theta_3=0 \qquad \omega=\alpha\ln|k\cdot x|-c\cdot x\end{aligned} \tag{8u}$$

$$\begin{aligned}&\theta_0=-\ln|k\cdot x| \qquad \theta_1=\tfrac{1}{2}b\cdot x(k\cdot x)^{-1} \qquad \theta_2=\tfrac{1}{2}c\cdot x(k\cdot x)^{-1}\\&\theta_3=\alpha\ln|k\cdot x| \qquad \omega=(a\cdot x)^2-(b\cdot x)^2-(c\cdot x)^2-(d\cdot x)^2.\end{aligned} \tag{8v}$$

where $a\cdot x$ stands for $a_\mu x^\mu$ and α is an arbitrary real constant.

We do not consider reduction of YMEs with the help of the above ansätze, because it is studied in great detail in [16].

We concentrate on the cases when the new (non-Lie) ansätze are obtained. It so happens that the procedure described gives rise to non-Lie ansätze provided the functions $\omega(x)$, $\theta_\mu(x)$ within the equivalence relations (7) have the form

$$\theta_\mu=\theta_\mu(\xi,\, b_\nu x^\nu,\, c_\nu x^\nu) \qquad \omega=\omega(\xi,\, b_\nu x^\nu,\, c_\nu x^\nu). \tag{9}$$

The list of inequivalent solutions of the system of PDEs (3), (4) satisfying (9) is exhausted by the following solutions:

$$\begin{aligned}&\theta_0=\theta_3=0 \qquad \omega=\tfrac{1}{2}k\cdot x\\&\theta_1=w_0(\xi)b\cdot x+w_1(\xi)c\cdot x \qquad \theta_2=w_2(\xi)b\cdot x+w_3(\xi)c\cdot x\end{aligned} \tag{10a}$$

$$\begin{aligned} &\omega = b\cdot x + w_1(\xi) \qquad \theta_0 = \alpha\Big(c\cdot x + w_2(\xi)\Big) \\ &\theta_a = -\tfrac{1}{4}\dot{w}_a(\xi) \quad a=1,2 \qquad \theta_3 = 0 \end{aligned} \tag{10b}$$

$$\begin{aligned} &\theta_0 = T(\xi) \qquad \theta_3 = w_1(\xi) \qquad \omega = b\cdot x\cos w_1 + c\cdot x \sin w_1 + w_2(\xi) \\ &\theta_1 = \Big(\tfrac{1}{4}(\varepsilon e^T + \dot{T})(b\cdot x\sin w_1 - c\cdot x\cos w_1) + w_3(\xi)\Big)\sin w_1 \\ &\qquad\qquad + \tfrac{1}{4}\Big(\dot{w}_1(b\cdot x\sin w_1 - c\cdot x\cos w_1) - \dot{w}_2\Big)\cos w_1 \\ &\theta_2 = -\Big(\tfrac{1}{4}(\varepsilon e^T + \dot{T})(b\cdot x\sin w_1 - c\cdot x\cos w_1) + w_3(\xi)\Big)\cos w_1 \\ &\qquad\qquad + \tfrac{1}{4}\Big(\dot{w}_1(b\cdot x\sin w_1 - c\cdot x\cos w_1) - \dot{w}_2\Big)\sin w_1 \end{aligned} \tag{10c}$$

$$\begin{aligned} &\theta_0 = 0 \qquad \theta_3 = \arctan\Big([c\cdot x + w_2(\xi)][b\cdot x + w_1(\xi)]^{-1}\Big) \\ &\theta_a = -\tfrac{1}{4}\dot{w}_a(\xi) \quad a=1,2 \qquad \omega = \Big([b\cdot x + w_1(\xi)]^2 + [c\cdot x + w_2(\xi)]^2\Big)^{1/2}. \end{aligned} \tag{10d}$$

Here $\alpha \neq 0$ is an arbitrary constant, $\varepsilon = \pm 1$, $w_0,\ w_1,\ w_2,\ w_3$ are arbitrary smooth functions on $\xi = \frac{1}{2}k\cdot x$ and $T = T(\xi)$ is a solution of the nonlinear ODE

$$(\dot{T} + \varepsilon e^T)^2 + \dot{w}_1^2 = \varkappa e^{2T} \qquad \varkappa \in \mathbb{R}^1 \tag{11}$$

where a dot over the symbol denotes differentiation with respect to ξ.

Substitution of the ansatz (2), where $R_{\mu\nu}(x)$ are given by expressions (3), (10), in the YMEs yields systems of nonlinear ODEs of the form (5), where

$$\begin{aligned} &k_{\mu\gamma} = -\tfrac{1}{4}k_\mu k_\gamma \qquad l_{\mu\gamma} = -(w_0 + w_3)k_\mu k_\gamma \\ &m_{\mu\gamma} = -4\ (w_0^2 + w_1^2 + w_2^2 + w_3^2)k_\mu k_\gamma - (\dot{w}_0 + \dot{w}_3)k_\mu k_\gamma \\ &q_{\mu\nu\gamma} = \tfrac{1}{2}(g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma) \\ &h_{\mu\nu\gamma} = (w_0 + w_3)(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) + 2(w_1 - w_2)\Big((k_\mu b_\nu - k_\nu b_\mu)\ c_\gamma \\ &\qquad\qquad + (b_\mu c_\nu - b_\nu c_\mu)k_\gamma + (c_\mu k_\nu - c_\nu k_\mu)b_\gamma\Big) \end{aligned} \tag{12a}$$

$$\begin{aligned} &k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma \qquad l_{\mu\gamma} = 0 \qquad m_{\mu\gamma} = -\alpha^2(a_\mu a_\gamma - d_\mu d_\gamma) \\ &q_{\mu\nu\gamma} = g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma \\ &h_{\mu\nu\gamma} = \alpha\Big((a_\mu d_\nu - a_\nu d_\mu)c_\gamma + (d_\mu c_\nu - d_\nu c_\mu)a_\gamma + (c_\mu a_\nu - c_\nu a_\mu)d_\gamma\Big) \end{aligned} \tag{12b}$$

$$\begin{aligned} &k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma \qquad l_{\mu\gamma} = -(\varepsilon/2)b_\mu k_\gamma \\ &m_{\mu\gamma} = -(\varkappa/4)k_\mu k_\gamma \qquad q_{\mu\nu\gamma} = g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma \\ &h_{\mu\nu\gamma} = (\varepsilon/4)(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) \end{aligned} \tag{12c}$$

$$\begin{aligned} &k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma \qquad l_{\mu\gamma} = -\omega^{-1}(g_{\mu\gamma} + b_\mu b_\gamma) \\ &m_{\mu\gamma} = -\omega^{-2}c_\mu c_\gamma \qquad q_{\mu\nu\gamma} = g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma \\ &h_{\mu\nu\gamma} = \tfrac{1}{2}\omega^{-1}(g_{\mu\gamma}b_\nu - g_{\mu\nu}b_\gamma). \end{aligned} \tag{12d}$$

3. Exact solutions of the nonlinear Yang–Mills equations

The systems (5), (12) are systems of twelve nonlinear second-order ODEs with variable coefficients. That is why there is little hope of constructing their general solutions. However, it is possible to obtain particular solutions of the system (5) whose coefficients are given by expressions (12*b*)–(12*d*).

Consider, as an example, the system of ODEs (5) with coefficients given by expressions (12*b*). We seek its solutions in the form

$$B_\mu = k_\mu e_1 f(\omega) + b_\mu e_2 g(\omega) \qquad fg \neq 0 \tag{13}$$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$.

On substituting the expression (13) in the above-mentioned system we get

$$\ddot{f} + (\alpha^2 - e^2 g^2) f = 0 \qquad f\dot{g} + 2\dot{f}g = 0. \tag{14}$$

The second ODE from (14) is easily integrated to give

$$g = \lambda f^{-2} \qquad \lambda \in \mathbb{R}^1 \quad \lambda \neq 0. \tag{15}$$

Substitution of the result obtained in the first ODE from (14) yields the Ermakov-type equation for $f(\omega)$:

$$\ddot{f} + \alpha^2 f - e^2\lambda^2 f^{-3} = 0,$$

which is integrated in elementary functions [14]

$$f = \left(\alpha^{-2}C^2 + \alpha^{-2}(C^4 - \alpha^2 e^2 \lambda^2)^{1/2} \sin 2|\alpha|\omega\right)^{1/2}. \tag{16}$$

Here $C \neq 0$ is an arbitrary constant.

Substituting (13), (15), (16) in the corresponding ansatz for $A_\mu(x)$, we get the following class of exact solutions of the YMEs (1):

$$\begin{aligned} A_\mu = {} & e_1 k_\mu \exp\,(-\alpha c\cdot x - \alpha w_2)\left(\alpha^{-2}C^2 + \alpha^{-2}(C^4 - \alpha^2 e^2\lambda^2)^{1/2} \sin 2|\alpha|(b\cdot x + w_1)\right)^{1/2} \\ & + e_2\lambda\left(\alpha^{-2}C^2 + \alpha^{-2}(C^4 - \alpha^2 e^2\lambda^2)^{1/2} \sin 2|\alpha|(b\cdot x + w_1)\right)^{-1} \\ & \times\left(b_\mu + \tfrac{1}{2}k_\mu \dot{w}_1\right). \end{aligned}$$

In a similar way we have obtained five other classes of exact solutions of the Yang–Mills equations:

$$\begin{aligned} A_\mu = {} & e_1 k_\mu e^{-T}(b\cdot x\cos w_1 + c\cdot x \sin w_1 + w_2)^{1/2} Z_{1/4}\Big((\mathrm{i}e\lambda/2)(b\cdot x\cos w_1 \\ & + c\cdot x\sin w_1 + w_2)^2\Big) + e_2\lambda\ (b\cdot x\cos w_1 + c\cdot x\sin w_1 + w_2) \\ & \times\Big(c_\mu\cos w_1 - b_\mu \sin w_1 + 2k_\mu[\tfrac{1}{4}(\varepsilon e^T + \dot{T})(b\cdot x\sin w_1 \\ & - c\cdot x\cos w_1) + w_3]\Big) \end{aligned}$$

$$\begin{aligned} A_\mu = {} & e_1 k_\mu e^{-T}\Big(C_1\cosh[e\lambda(b\cdot x\cos w_1 + c\cdot x\sin w_1 + w_2)] + C_2\sinh[e\lambda \\ & \times(b\cdot x\cos w_1 + c\cdot x\sin w_1 + w_2)]\Big) + e_2\lambda\Big(c_\mu\cos w_1 - b_\mu\sin w_1 \\ & + 2k_\mu[\tfrac{1}{4}(\varepsilon e^T + \dot{T})(b\cdot x\sin w_1 - c\cdot x\cos w_1) + w_3]\Big); \end{aligned}$$

$$A_\mu = e_1 k_\mu e^{-T}\Big(C^2(b\cdot x\cos w_1 + c\cdot x\sin w_1 + w_2)^2 + \lambda^2 e^2 C^{-2}\Big)^{1/2}$$
$$+e_2\lambda\Big(C^2(b\cdot x\cos w_1 + c\cdot x\sin w_1 + w_2)^2 + \lambda^2 e^2 C^{-2}\Big)^{-1}$$
$$\times\Big(b_\mu\cos w_1 + c_\mu\sin w_1 - \tfrac{1}{2}k_\mu[\dot w_1(b\cdot x\sin w_1 - c\cdot x\cos w_1) - \dot w_2]\Big)$$
$$A_\mu = e_1 k_\mu Z_0\Big((\mathrm{i}e\lambda/2)[(b\cdot x + w_1)^2 + (c\cdot x + w_2)^2]\Big) + e_2\lambda\Big(c_\mu(b\cdot x + w_1)$$
$$-b_\mu(c\cdot x + w_2) - \tfrac{1}{2}k_\mu[\dot w_1(c\cdot x + w_2) - \dot w_2(b\cdot x + w_1)]\Big);$$
$$A_\mu = e_1 k_\mu\Big(C_1[(b\cdot x + w_1)^2 + (c\cdot x + w_2)^2]^{e\lambda/2} + C_2[(b\cdot x + w_1)^2$$
$$+(c\cdot x + w_2)^2]^{-e\lambda/2}\Big) + e_2\lambda[(b\cdot x + w_1)^2 + (c\cdot x + w_2)^2]^{-1}$$
$$\times\Big(c_\mu(b\cdot x + w_1) - b_\mu(c\cdot x + w_2) - \tfrac{1}{2}k_\mu[\dot w_1(c\cdot x + w_2)$$
$$-\dot w_2(b\cdot x + w_1)]\Big).$$

Here $C_1,\ C_2,\ C \neq 0, \lambda$ are arbitrary parameters, $w_1,\ w_2,\ w_3$ are arbitrary smooth functions on $\xi = \frac{1}{2}k\cdot x$ and $T = T(\xi)$ is a solution of the ODE (11). In addition, we use the following notation:

$$k\cdot x = k_\mu x^\mu \qquad b\cdot x = b_\mu x^\mu \qquad c\cdot x = c_\mu x^\mu$$
$$Z_s(\omega) = C_1 J_s(\omega) + C_2 Y_s(\omega)$$
$$e_1 = (1,0,0) \qquad e_2 = (0,1,0)$$

where $J_s,\ Y_s$ are the Bessel functions.

Thus we have obtained broad families of exact non-Abelian solutions of the YMEs (1). It can be verified by direct and rather involved computation that the solutions obtained are not self-dual, i.e. that they do not satisfy self-dual YMEs.

4. Conclusion

Let us say a few words about symmetry interpretation of the ansätze (2), (3), (10). Consider, as an example, the ansatz determined by expressions (10a). As a direct computation shows, generators of a three-parameter Lie group G leaving it invariant are of the form

$$Q_1 = k_\alpha\partial_\alpha$$
$$Q_2 = b_\alpha\partial_\alpha - 2[w_0(k_\mu b_\nu - k_\nu b_\mu) + w_2(k_\mu c_\nu - k_\nu c_\mu)]\sum_{a=1}^{3} A^{a\nu}\partial_{A^{a\mu}} \tag{17}$$
$$Q_3 = c_\alpha\partial_\alpha - 2[w_1(k_\mu b_\nu - k_\nu b_\mu) + w_3(k_\mu c_\nu - k_\nu c_\mu)]\sum_{a=1}^{3} A^{a\nu}\partial_{A^{a\mu}}.$$

Evidently, the system of PDEs (1) is invariant under the one-parameter group G_1 having the generator Q_1. But it is not invariant under the groups having the generators $Q_2,\ Q_3$. Consider, as an example, the generator Q_2. Acting by the second prolongation of the operator Q_2 (which is constructed in a standard way, see, e.g., [18, 20]) on the system of PDEs (1), after some tedious algebra we obtain the following equality:

$$\begin{aligned}
\underset{2}{Q_2}L_\mu = {} & 2\Big(w_0(k_\mu b_\nu - k_\nu b_\mu) + w_2(k_\mu c_\nu - k_\nu c_\mu)\Big)L^\nu \\
& +2\Big(\dot{w}_0(k_\mu b_\nu - k_\nu b_\mu) + \dot{w}_2(k_\mu c_\nu - k_\nu c_\mu)\Big)\underline{Q_1 A^\nu} \\
& -\partial^\mu\Big((w_0 b_\nu + w_2 c_\nu)\underline{Q_1 A^\nu} - k_\nu(w_0\underline{Q_2 A^\nu} + w_2\underline{Q_3 A^\nu})\Big) \\
& -(w_0 b_\mu + w_2 c_\mu)\partial_\nu \underline{Q_1 A_\nu} - k_\mu\Big(w_0(w_0 b_\nu + w_2 c_\nu) \\
& +w_2(w_1 b_\nu + w_3 c_\nu)\Big)\underline{Q_1 A^\nu} \\
& +e\Big((w_0 b_\nu + w_2 c_\nu)\underline{Q_1 A^\nu} - k_\nu(w_0\underline{Q_2 A^\nu} + w_2\underline{Q_3 A^\nu})\Big) \times A_\mu \\
& +2e(w_0 b_\nu A^\nu + w_2 c_\nu A^\nu) \times \underline{Q_1 A_\mu} \\
& -2ek_\nu A^\nu \times (w_0\underline{Q_2 A_\mu} + w_2\underline{Q_3 A_\mu}) + eA_\nu \times (w_0 b_\mu + w_2 c_\mu)\underline{Q_1 A^\nu} \\
& -ek_\mu A_\nu \times (w_0\underline{Q_2 A^\nu} + w_2\underline{Q_3 A^\nu}).
\end{aligned} \tag{18}$$

In the above expressions we use the designations

$$\begin{aligned}
L_\mu \equiv {} & \partial_\nu\partial^\nu A_\mu - \partial^\mu\partial_\nu A_\nu + e\Big((\partial_\nu A_\nu) \times A_\mu - 2(\partial_\nu A_\mu) \times A_\nu + (\partial^\mu A_\nu) \times A^\nu\Big) \\
& +e^2 A_\nu \times (A^\nu \times A_\mu)
\end{aligned}$$

$$Q_1 A_\mu \equiv k_\alpha \partial_\alpha A_\mu$$

$$Q_2 A_\mu \equiv b_\alpha \partial_\alpha A_\mu + 2\Big(w_0(k_\mu b_\nu - k_\nu b_\mu) + w_2(k_\mu c_\nu - k_\nu c_\mu)\Big)A^\nu$$

$$Q_3 A_\mu \equiv c_\alpha \partial_\alpha A_\mu + 2\Big(w_1(k_\mu b_\nu - k_\nu b_\mu) + w_3(k_\mu c_\nu - k_\nu c_\mu)\Big)A^\nu$$

and by the symbol $\underset{2}{Q_2}$ we denote the second prolongation of the operator Q_2.

As the underlined terms in (18) do not vanish on the set of solutions of the YMEs, the system of PDEs (1) is not invariant under the Lie transformation group G_2 having the generator Q_2. On the other hand, the system

$$L_\mu = 0 \qquad Q_a A_\mu = 0 \quad a = 1, 2, 3$$

is evidently invariant under the group G_2. The same assertion holds for the Lie transformation group G_3 having the generator Q_3. Consequently, the YMEs are conditionally invariant with respect to the three-parameter Lie transformation group $G = G_1 \otimes G_2 \otimes G_3$. This means that solutions of the YMEs obtained with the help of the ansatz invariant under the group with generators (17) cannot be found by means of the classical symmetry reduction procedure.

As rather tedious computations show, the ansätze determined by expressions (10*b*)–(10*d*) also correspond to conditional symmetry of the YMEs. Hence it follows, in particular, that the YMEs should be included in the long list of mathematical and theoretical physics equations possessing non-trivial conditional symmetry [7].

Another interesting observation is that by specifying the arbitrary functions contained in non-Lie ansätze in an appropriate way, one can obtain some Lie ansätze. Really, expressions (3), (8*l*), (8*m*), (8*q*) are particular cases of expressions (10*a*), expressions

(8*a*), (8*e*), (8*f*), (8*g*), (8*n*), (8*o*), (8*s*), (8*t*), (8*u*) are particular cases of expressions (10*b*), (10*c*) and expressions (8*h*), (8*p*) are particular cases of the expressions (10*d*). So if we denote the invariant solutions of the Yang–Mills equations symbolically by the dots in some space of solutions of the system of PDEs (1), then some of them can be connected by curves which are *conditionally-invariant* solutions! Thus, the at first glance distinct solutions are particular cases of more general solutions. A similar assertion holds for the nonlinear wave [13] and Dirac [11] equations. On the other hand, some invariant solutions (namely those determined by expressions (8*b*), (8*d*), (8*i*), (8*j*), (8*k*), (8*r*), (8*v*)) cannot be connected with other solutions by the curve that is a conditionally-invariant solution of the form (10). A possible explanation of this fact is that there exist more general conditionally-invariant solutions of YMEs.

The above picture admits an analogy with the case where the equation under study has a general solution. In that case, each two solutions can be connected by a curve which is a solution of the equation. The only exceptions are the singular solutions which are obtained by some asymptotic procedure. So one can guess that there exists a collection of conditionally-invariant solutions of YMEs such that the majority of invariant solutions are their particular cases and the remaining ones are obtained from these by an asymptotic procedure. However, this problem so far remains completely open and needs further investigation.

Our last remark is that the procedure suggested here also yields some well known exact solutions of YMEs. For example, the ansatz for the Yang–Mills field determined by expressions (2), (3) and (8*v*) gives rise to the meron and instanton solutions of the system (1), originally obtained with the help of the ansatz suggested by 't Hooft [21], Corrigan and Fairlie [5] and Wilczek [22] (for more details, see [16]).

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